Elementary Properties of Clock-regulated Queues

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Abstract

A single clock-regulated queue is fed by Bernouilli streams of messages. We obtain the distributions of queue length, of waiting time, and of output for such a system. This is generalized as well to the case of a queue of finite capacity.

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1. Introduction

The classical theory of networks of interconnected queues has reached a respectable level of maturity. Steady state problems, in particular, yield to well-established routines [6]. However, the classical theory, in which arrival times cover a continuous domain, is not appropriate to the clock–regulated arrivals that are a necessary component of networks encountered in current and anticipated large scale digital computers. In such systems, there is a common unit time interval between arrivals and between departures from the queue. This paper addresses issues that arose in assessing the operation of the NYU Ultracomputer [2], that are typical of the generation that it represents, and that correspond to rather direct but nontrivial extension of queuing theory to networks with quantized time. In particular, we will ask how an i.i.d. input stream to a single server is transformed to variable queue length and output streams, a problem which will be conveniently represented as a random walk on a suitably bounded one-dimensional lattice.

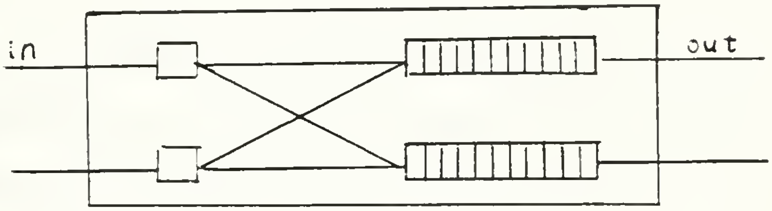

Fig. 1. 2 × 2 buffered switch

The basic building block of such an interconnected network, as described in [3], is a 2 × 2 buffered switch. Each input port can accept an information packet per unit time and route it to the appropriate output port. Each output port has a FIFO buffer. Conflicts between messages simultaneously routed to the same output port are resolved by queuing the messages. Hence, we represent a 2 × 2 switch as a system of two queues working in parallel, each one with a deterministic server (of service time equal to 1). Any message entering either of the two inputs of the switch goes with probability 1/2 to either of the two output queues of the switch. At this first stage, arriving messages are i.i.d. random variables.

Kruskal et al. [3] idealize this structure and assume that each output port has an infinite capacity, i.e. can accept any number of messages from the input port. Using this assumption, they find the generating function for the steady state distribution of waiting times (at this first stage of the network) for each output port. While this idealization of infinite capacity is clearly infeasible in practice, it provides a reasonable approximation for small p (probability of arrival) and moderate capacity.

Under the above assumptions, we will develope here relations for the exact probability distribution of the number of waiting messages (for both finite and infinite capacity output port) at the end of the n^{th} input cycle. These relations will be given in the form of difference equations for the probability distribution of the number of waiting messages at each output port. The single queue or one dimensional distribution will be solved completely. We will obtain the probability distribution for the number of waiting messages in steady state for both finite and infinite capacity output port. We will also find the probability that the queue is empty for the first time at the end of the nth cycle.

Another problem when dealing with more than one stage of the network is that the output distribution, which becomes the input for a second stage, no longer consists of i.i.d. random variables. We will show that in steady state, the lengths of the intervals between zero output are independent random variables and find their probability distributions.

2.The Infinite Capacity Queue as a Random Walk on the Half Line

Although the buffered switch encompasses two connected queues, the connections are only at the input, so that each queue is unaffected by the operation of the other. It is true that the outputs are correlated in a nontrivial fashion, but in the networks of interest [2], the two output streams will never feed directly to the same switch. Thus, we may immediately convert the problem to that of a single queue, with a specially designed input stream.

Let us start by obtaining the statistics of the number of waiting messages when the queue in question has infinite capacity. Our assumptions are then as follows:

i) At the (first stage) input, arriving messages are treated as i.i.d. Bernoulli variables: at each cycle, a message arrives with probability p and enters one of the two output queues with probability 1/2. Hence, the probability that j messages arrive in any cycle at a given queue is

$$f_j = \binom{2}{j}\left(\frac{p}{2}\right)^j\left(1-\frac{p}{2}\right)^{2-j} \quad j = 0, 1, 2. \tag{2.1}$$

ii) Every message requires one time unit (cycle) to be served.

iii) If S_n is the number of waiting messages at the end of the n^{th} cycle, and a_n the number of messages arriving at the beginning of the n^{th} cycle, then

$$\begin{aligned} S_0 &= 0 \\ S_n &= \max\{0, S_{n-1} + a_n - 1\} \quad \text{for} \quad n \geq 1, \end{aligned} \tag{2.2}$$

i.e. a single message leaves during the nth cycle if at least one has appeared on the queue.

We can now easily convert to a random walk interpretation. Consider a particle executing a random walk, starting at the origin, on the infinite lattice of nonnegative integers. When at the origin, the particle moves a unit step to the right with probability $f_2 = p^2/4$, or stays at the origin with probability $f_0 + f_1 = 1 - p^2/4$. From position $j \geq 1$, the particle moves a unit step to the right with probability $f_2 = p^2/4$, a unit step to the left with probability $f_0 = (1 - p/2)^2$, or stays at j with probability $f_1 = p(1 - p/2)$. Then

$p_n(j)$, the probability of being at position j at the nth step, represents $P(S_n = j)$, the probability of having j messages in the queue at the nth cycle, and clearly satisfies

$$\begin{aligned} &p_o(j) = \delta_{j,0} \\ (2.3) \qquad &p_n(0) = f_0 p_{n-1}(1) + (f_0 + f_1) p_{n-1}(0) \quad \text{for} \quad n > 0 \\ &p_n(j) = f_0 p_{n-1}(j+1) + f_1 p_{n-1}(j) + f_2 p_{n-1}(j-1) \quad \text{for} \quad n > 0,\ j \geq 1. \end{aligned}$$

Eqs. (2.3) are most easily solved by introducing an appropriate generating function. We first observe that $\sum_{j=0}^{\infty} p_n(j) x^j$ is bounded by 1, and hence analytic, for $|x| \leq 1$ in the complex plane: it follows that

$$(2.4) \qquad F(x, \lambda) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} p_n(j) x^j \lambda^n$$

is analytic for $|\lambda| < 1$ as well. Now from (2.3) we have, on carrying out the sum indicated in (2.4),

$$(2.5) \qquad \begin{gathered} F(x, \lambda)(x - \lambda R(x)) = x + \lambda(1 - \frac{p}{2})^2 F(0, \lambda)(x - 1) \\ \text{where} \quad R(x) = (1 - \frac{p}{2} + \frac{p}{2} x)^2. \end{gathered}$$

The equation $x - \lambda R(x) = 0$ has two roots:

$$(2.6) \qquad \lambda(\frac{p}{2})^2 x_i(\lambda)^2 + (\lambda p(1 - \frac{p}{2}) - 1) x_i(\lambda) + \lambda(1 - \frac{p}{2})^2 = 0, \quad i = 0, 1$$

one of which,

$$(2.7) \qquad x_0(\lambda) = (1 - \lambda p(1 - \frac{p}{2}) - \sqrt{1 - 2\lambda p(1 - \frac{p}{2})})/(\frac{p^2}{2}\lambda),$$

satisfies $|x_0(\lambda)| < 1$ on the branch for which $x_0(\lambda) \to 0$ as $\lambda \to 0$. Hence, choosing $x = x_0(\lambda)$ in (2.5), we have

$$(2.8) \qquad \lambda(1 - \frac{p}{2})^2 F(0, \lambda) = \frac{x_0(\lambda)}{1 - x_0(\lambda)},$$

and consequently

$$(2.9) \qquad F(x, \lambda) = \frac{x - \frac{1-x}{1-x_0(\lambda)} x_0(\lambda)}{x - \lambda R(x)}.$$

Eqs. (2.6) and (2.8), coupled with the Lagrange implicit function expansion [5]:

$$\text{if} \quad x = \lambda\phi(x), \quad \text{then}$$

(2.10)
$$K(x) = K(0) + \sum_{1}^{\infty} \frac{\lambda^n}{n!}\left(\frac{d}{dx}\right)^{n-1}\left(\phi(x)^n K'(x)\right)\Big|_{x=0},$$

enable us to find the coefficient of λ^n in $F(0, \lambda)$ and hence the probability of returning to the origin at the nth step (i.e. the probability of the queue being empty), resulting in the expression

(2.11)
$$p_n(0) = 1 - \frac{p/2}{1-p/2}\sum_{k=1}^{n} \frac{\binom{2k}{k}}{k+1}\left(\frac{p}{2}\left(1-\frac{p}{2}\right)\right)^k \quad \text{for} \quad n \geq 1.$$

In the special case $p = 1$, one has a symmetric random walk reflected at the origin, and (2.11) sums to the expected

(2.12)
$$\begin{aligned} p_n(0) &= 2\binom{2n+2}{n+1}/2^{2n+2} \\ &\to 2/((n+1)\pi)^{1/2} \quad \text{at large } n. \end{aligned}$$

Similarly, using (2.6) and (2.9), a more complicated expression for $p_n(j)$ can be obtained for all $j \geq 1$. The standard statistical paramters, however, are found quite simply. Consider for example $E(S_n)$, the mean number of messages in the queue at the end of the nth cycle. Since

(2.13)
$$\frac{\partial}{\partial x}F(x, \lambda)\Big|_{x=1} = \sum_{n=0}^{\infty} E(S_n)\lambda^n,$$

we have at once from (2.9)

(2.14)
$$E(S_n) = \sum_{k=0}^{n-1}\left(1-\frac{p}{2}\right)^2 p_k(0) - n(1-p),$$

where $p_k(0)$ is given by (2.11).

The steady state limiting distribution is of particular interest. If

(2.15)
$$p(j) = \lim_{n \to \infty} p_n(j).$$

then

$$F(x) = \sum_{j=0}^{\infty} \lim_{n \to \infty} p_n(j) x^j \tag{2.16}$$
$$= \lim_{n \to 1} (1-\lambda) F(x,\lambda) = \frac{(x-1)(1-p)}{x - R(x)},$$

which expands out without difficulty to yield

$$p(j) = \frac{1-p}{(1-\frac{p}{2})^2} \left(\frac{p/2}{1-p/2}\right)^{2j} \tag{2.17}$$

$$\lim_{n \to \infty} E(S_n) = p^2/4(1-p) \tag{2.18}$$

$$\lim_{n \to \infty} Var(S_n) = p^4/16(1-p)^2 \tag{2.19}$$

(2.18) and (2.19) were previously obtained [3]. Note that for p = 1,

$$F(0,\lambda) = \frac{2}{\lambda}((1-\lambda)^{-1/2} - 1), \ R(x) = \frac{1}{4}(1+x)^2, \tag{2.20}$$
$$F(x,\lambda) = \frac{1}{2}(1 + x - \frac{1-x}{\sqrt{1-\lambda}})/(x - \frac{\lambda}{4}(1+x)^2),$$

so that $\lim_{\lambda - 1}(1-\lambda)F(x,\lambda) = 0$, and there is no limiting distribution. However, it is easily verified in this case that

$$\lim_{n \to \infty} E(S_n)/n^{1/2} = 1/2\sqrt{\pi} \ \text{ at } \ p = 1. \tag{2.21}$$

Another quantity of interest (for $p < 1$) is the steady state distribution of the waiting time, which can be obtained directly from (2.17). Consider first $w(0)$, the probability of waiting zero cycles to be processed in the queue, hence, the queue must be empty, with probability $p(0) = (1-p)/(1-p/2)^2$. Further, either there is no other simultaneous arrival, probability $1-p/2$, or there is, but the message in question is chosen first to be processed. probability $\frac{1}{2}p/2$. Hence $w(0) = (1-p/2+\frac{1}{2}p/2)p(0)$:

(2.22) $$w(0) = \frac{1-p}{(1-\frac{p}{2})^2}(1-\frac{p}{4}).$$

As for $w(j)$, either the queue has length j and the message is taken first, or the length is $j-1$ and it is taken second. Together, then, $w(j) = (1-\frac{p}{4})p(j) + \frac{p}{4}p(j-1)$:

(2.23) $$w(j) = \frac{1-p}{(1-\frac{p}{2})^2}(\frac{p/2}{1-p/2})^{2j}\frac{1}{p} \quad \text{for} \quad j > 0.$$

It readily follows from (2.16) that

(2.24) $$W(x) = \sum_0^\infty w(j)x^j = \frac{1-p}{p}\frac{1-R(x)}{R(x)-x}.$$

The average waiting time is thus

(2.25) $$\frac{dW(x)}{dx}|_{x=1} = \frac{p}{4(1-p)}.$$

The fact that this is identical to the mean queue length (2.18) divided by the mean arrival rate p is an immediate consequence of Little's formula [4]. However, (2.24) allows all of the moments to be found just as easily.

3. The Output Process of the Queue

A queue transforms an input time process into an output time process. Since the i.i.d. input assumption proliferates throughout the literature, it is of interest to find the characteristics of the associated output, which of course serves in part as input to the next stage. As a first step, let us examine the probability g_n of returning to the origin for the first time at the nth step (i.e. the probability that the queue is empty for the first time at the nth cycle). Since

$$p_n(0) = g_n + p_1(0)g_{n-1} + \cdots + p_{n-1}(0)g_1, \tag{3.1}$$

and $F(0,\lambda) - 1 = \sum_{n=1}^{\infty} p_n(0)\lambda^n$ from (2.4), we obtain on summing, the interrelation

$$G(\lambda) = \sum_{i=1}^{\infty} g_i \lambda^i = (F(0,\lambda) - 1)/F(0,\lambda), \tag{3.2}$$

or inserting (2.8),

$$G(\lambda) = \lambda(1 - \frac{p^2}{4}) + \lambda\frac{p^2}{4}x_0(\lambda). \tag{3.3}$$

From (2.7), we have at once

$$\begin{aligned} g_1 &= 1 - \frac{p^2}{4} \\ g_n &= \frac{1}{n}\binom{2n-2}{n-1}(\frac{p}{2}(1-\frac{p}{2}))^n \quad \text{for} \quad n \geq 2. \end{aligned} \tag{3.4}$$

A rather direct consequence of (3.3) is that

$$\sum_{n=1}^{\infty} n g_n = (1 - \frac{p}{2})^2/(1-p) \tag{3.5}$$

for the mean number of steps until the first return to the origin. Note that $\sum_{n=1}^{\infty} n g_n = 1/\lim_{n\to\infty} p_n(0)$, which is an immediate result of Feller's Theorem [1].

Our principal interest lies in the output time series $\{O_n\}$ which according to our rules, satisfies

$$O_n = \begin{cases} 1 & if \quad S_{n-1} + a_n \geq 1 \\ 0 & \quad\;\; S_{n-1} + a_n = 0 \end{cases} \tag{3.6}$$

Since the event that the queue is empty for the first time at the $n+1^{th}$ cycle is not enough to enforce output at the first n cycles, we need to find explicitly the probability of no output for the first time at the $n+1^{th}$ cycle. A two-dimensional representation of the process is helpful, in which each vertical line at n signifies the transition from S_n to S_{n+1}. It shows that $\{O_n\}$ is not fully determined by $\{S_n\}$ because, when $S_n = 0$, the inputs $a_{n+1} = 1$

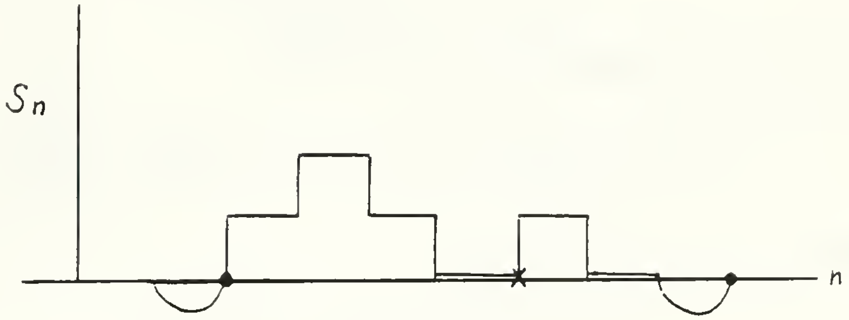

Fig. 2. Two-dimensional representation of state of queue

– indicated by a horizontal line, and $a_{n+1} = 0$ – indicated by a semicircle, both imply $S_{n+1} = 0$. Furthermore, $O_{n+1} = 0$ in the latter case, and only then. Since the terminus of such a semicircle always corresponds to $S_{n+1} = 0$, $a_{n+1} = 0$, the output describes a homogeneous renewal process, completely specified by the probability h_n that $O_{n+1} = 0$ for the first time, given that $O_0 = 0$.

Let us now relate the generating function

$$\overline{H}(\lambda) = \sum_{i=1}^{\infty} h_i \lambda^i \tag{3.7}$$

to $G(\lambda)$ computed above. To do so, we decompose any trajectory of h_n — a segment between two successive termini of semicircles — into a first arrival at $s_j = 0$ (an x in Fig. 2), followed by a segment from h_{n-j}. The former is a trajectory of g_j except when $j = 1$, in which case the possibility of an $O_1 = 0$ semicircle must be excluded. If the corresponding trajectories are denoted by $\overline{g}_j$, then

$$\begin{aligned} \overline{g}_j &= g_j \quad \text{for} \quad j > 1 \\ \overline{g}_1 &= p(1 - \frac{p}{2}), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \overline{G}(\lambda) &= \sum_1^\infty \overline{g}_j \lambda^j \\ &= \lambda p(1 - \frac{p}{2}) + \lambda \frac{p^2}{4} x_0(\lambda). \end{aligned} \tag{3.9}$$

With the above notation, it is now clear that

$$h_n = (1 - \frac{p}{2})^2 \overline{g}_n + \overline{g}_{n-1} h_1 + \cdots + \overline{g}_1 h_{n-1}, \quad n \geq 1. \tag{3.10}$$

Multiplying by λ^n and summing, then,

$$\overline{H}(\lambda) = (1 - \frac{p}{2})^2 \overline{G}(\lambda) + \overline{G}(\lambda)\overline{H}(\lambda) \tag{3.11}$$

Since $h_0 = (1 - \frac{p}{2})^2$, we conclude that

$$H(\lambda) = h_0 + \overline{H}(\lambda) = \frac{(1 - \frac{p}{2})^2}{1 - \lambda p(1 - \frac{p}{2}) - \lambda \frac{p^2}{4} x_0(\lambda)}, \tag{3.12}$$

or on using (2.6), simply

$$H(\lambda) = \frac{1}{\lambda} x_0(\lambda) \tag{3.13}$$

Direct expansion of (3.13) yields

$$h_n = (1 - \frac{p}{2})^2 \frac{\binom{2n+2}{n+1}}{n+2} (\frac{p}{2}(1 - \frac{p}{2}))^n. \tag{3.14}$$

This is to be contrasted with each input stream, also representing a steady state current of p messages per unit time, in which the i.i.d. character produces instead a probability for an n–output cluster of

$$h_n^0 = (1 - p)p^n. \tag{3.15}$$

Since the currents are the same, the mean cluster size $\frac{p^2}{1-p^2}$ will be the same for (3.14) and (3.15), but the variance will not be. One readily finds that

$$\begin{aligned} Var(n; h_n) &= \frac{p}{(1-p)^2} + \frac{p^2/2}{(1-p)^3} \\ Var(n; h_n^0) &= \frac{p}{(1-p)^2}, \end{aligned} \tag{3.16}$$

the ratio

$$\frac{Var(n; h_n)}{Var(n; h_n^0)} = 1 + \frac{\frac{p}{2}}{1-p} \tag{3.17}$$

departing substantially from 1 as $p \to 1$: the output becomes much more skewed toward large clusters than the input.

4. The Finite Capacity Queue.

Until now, we dealt with a random walk on the infinite half–lattice, which corresponds to a queue with infinite capacity. This idealization is clearly not feasible in practice, although it closely approximates the correct performance for small probability of arrival and moderate capacity. We will therefore investigate now the finite (N) capacity queue which again resembles a random walk, but now on a bounded lattice with reflecting boundary at N. We will present two possible reflection models for the finite capacity queue.

$$ModelI : S_n = min\{N, max(0, S_{n+1} + a_n - 1)\} \tag{4.1}$$

$$ModelII : S_n = \begin{cases} max\{0, S_{n-1} + a_n - 1\} & \text{if} \quad S_{n-1} \leq N - 1 \\ N - 1 & \text{if} \quad S_{n-1} = N \end{cases} \tag{4.2}$$

The two models differ only in their boundary conditions, the first corresponding to a simple reflection of the walk by a barrier placed at $N + \frac{1}{2}$, while the second represents a more realistic network operation in which no message is accepted when the queue is saturated. We will present the relevant difference equations for both models, solve model I systematically, and quote the analogous results for model II in steady state.

For model I, proceeding as in Sec. 2, $p_n(j)$ satisfies the following:

$$\begin{aligned} &p_o(0) = 1 \\ &p_n(0) = f_0 p_{n-1}(1) + f_1 p_{n-1}(0) + f_0 p_{n-1}(0) \\ &p_n(j) = f_0 p_{n-1}(j+1) + f_1 p_{n-1}(j) + f_2 p_{n-1}(j-1) \quad \text{for} \quad 1 \leq j \leq N-1 \\ &p_n(N) = f_2 p_{n-1}(N) + f_1 p_{n-1}(N) + f_2 p_{n-1}(N-1). \end{aligned} \tag{4.3}$$

Hence, writing

$$\begin{aligned} F_n^N(x) &\equiv \sum_{j=0}^{N} p_n(j) x^j \\ F^N(x, \lambda) &\equiv \sum_{n=0}^{\infty} F_n^N(x) \lambda^n , \end{aligned} \tag{4.4}$$

we have at once from (4.3)

$$F^N(x,\lambda)[x-\lambda R(x)] = x + \lambda f_0 \sum_{n=0}^{\infty} p_n(0)\lambda^n(x-1) \tag{4.5}$$
$$- \lambda f_2 \sum_{n=0}^{\infty} p_n(N)\lambda^n x^{N+1}(x-1).$$

But if $x_0(\lambda)$ and $x_1(\lambda)$ are the two roots of $x - \lambda R(x) = 0$, then from (4.5),

$$\begin{aligned} 0 &= x_0(\lambda) + \lambda f_0(x_0(\lambda)-1)\sum_{n=0}^{\infty} p_n(0)\lambda^n \\ &\quad - \lambda f_2 \sum_{n=0}^{\infty} p_n(N)\lambda^n(x_0(\lambda)-1)x_0(\lambda)^{N+1} \\ 0 &= x_1(\lambda) + \lambda f_0(x_1(\lambda)-1)\sum_{n=0}^{\infty} p_n(0)\lambda^n \\ &\quad - \lambda f_2 \sum_{n=0}^{\infty} p_n(N)\lambda^n(x_1(\lambda)-1)x_1(\lambda)^{N+1}. \end{aligned} \tag{4.6}$$

It follows that

$$\begin{aligned} \lambda f_0 \sum_{n=0}^{\infty} p_n(0)\lambda^n &= \frac{\lambda f_0}{\lambda-1}\frac{x_1(\lambda)^N - x_0(\lambda)^N - (x_1(\lambda)^{N-1} - x_0(\lambda)^{N-1})}{x_0(\lambda)^{N+1} - x_1(\lambda)^{N+1}} \\ \lambda f_2 \sum_{n=0}^{\infty} p_n(N)\lambda^n &= \frac{\lambda f_2}{\lambda-1}\frac{x_1(\lambda)-x_0(\lambda)}{x_0(\lambda)^{N+1}-x_1(\lambda)^{N+1}} \\ \text{where }\ x_0(\lambda), x_1(\lambda) &= [1-\lambda p(1-\frac{p}{2}) \mp \sqrt{1-2\lambda p(1-\frac{p}{2})}]/\frac{p^2}{2}\lambda. \end{aligned} \tag{4.7}$$

and using (4.5), we obtain

(4.8)
$$\begin{aligned} &F^N(x,\lambda) \\ &= \frac{1}{x-\lambda R(x)}\{x + \frac{\lambda(x-1)}{1-\lambda}[(1-\frac{p}{2})^2(\frac{x_1(\lambda)^N - x_0(\lambda)^N - (x_1(\lambda)^{N-1} - x_0(\lambda)^{N-1}}{x_1(\lambda)^{N+1} - x_0(\lambda)^{N+1}}) \\ &\quad - \frac{p^2}{4}x^{N+1}(\frac{x_1(\lambda)-x_0(\lambda)}{x_1(\lambda)^{N+1}-x_0(\lambda)^{N+1}})]\}. \end{aligned}$$

Eq. (4.8) enables us to find the moments of the time-dependent distribution. e.g. $E(S_n)$ as in (2.13). or the second moment via $\sum \lambda^n E[S_n(S_n-1)] = \partial^2 F^N(x.\lambda)/\partial x^2|_{x=1}$.

etc. Again however, our major interest will be in the steady state properties of the process, obtained as in (2.15) and (2.16). We find without difficulty from (4.8) that

$$F^N(x) = \lim_{\lambda \to 1}(1-\lambda)F^N(x,\lambda) = \frac{1-\beta}{1-\beta x}\,\frac{1-\beta^{N+1}x^{N+1}}{1-\beta^{N+1}} \qquad \text{where} \quad \beta = \left(\frac{p/2}{1-p/2}\right)^2, \tag{4.9}$$

leading to

$$p(j) = \frac{1-\beta}{1-\beta^{N+1}}\beta^j \quad \text{for} \quad j = 0, 1, \ldots, N, \tag{4.10}$$

to be compared with (2.17). It follows that

$$E[\lim_{n\to\infty} S_n] = \sum_{j=0}^{N} jp(j) = \frac{\beta}{1-\beta}\{1 - \frac{(N+1)(1-\beta)\beta^N}{1-\beta^{N+1}}\}, \tag{4.11}$$

with a more complicated expression for $Var[\lim_{n\to\infty} S_n]$. However, specialization to $p = 1$ is very simple indeed:

$$\begin{gathered} \text{if} \quad p = 1, \quad \text{then} \quad p(j) = \frac{1}{N+1}, \quad j = 1, \ldots, N \\ E[\lim_{n\to\infty} S_n] = N/2, \quad Var[\lim_{n\to\infty} S_n] = N(N+2)/12. \end{gathered} \tag{4.12}$$

Model II proceeds in quite the same fashion as Model I. Here the difference equations take the form

$$\begin{aligned} p_n(0) &= f_0 p_{n-1}(1) + f_1 p_{n-1}(0) + f_0 p_{n-1}(0) \\ p_n(k) &= f_0 p_{n-1}(k+1) + f_1 p_{n-1}(k) + f_2 p_{n-1}(k-1) \quad \text{for} \quad 1 \le k \le N-2 \\ p_n(N-1) &= p_{n-1}(N) + f_1 p_{n-1}(N-1) + f_2 p_{n-1}(N-2) \\ p_n(N) &= \qquad\qquad\qquad\qquad f_2 p_{n-1}(N-1). \end{aligned} \tag{4.13}$$

and lead directly as in Model I to the steady state distribution

$$p(j) = \lim_{n\to\infty} p_n(j) = \begin{cases} \frac{1-\beta}{1-\beta^N p}\beta^j & \text{for} \quad j = 0,\ldots, N-1 \\ \frac{1-\beta}{1-\beta^N p}(1-\frac{p}{2})^2\beta^N & \text{for} \quad j = N \end{cases} \tag{4.14}$$

for $p < 1$, whereas for $p = 1$,

$$p(j) = \begin{cases} \frac{4}{4N+1} & \text{for} \quad j = 0,\ldots, N-1 \\ \frac{1}{4N+1} & \text{for} \quad j = N \end{cases} \tag{4.15}$$

Consequently, one finds

$$E[\lim_{n\to\infty} S_n) = \begin{cases} \frac{\beta}{1-\beta}(1 - \frac{(1-p)\beta^N}{1-\beta^N p}) - \frac{N\beta^N p}{1-\beta^N p} & \text{for} \quad p < 1 \\ \frac{N}{2}(\frac{4N-2}{4N+1}) & \text{for} \quad p = 1 \end{cases} \tag{4.16}$$

5. Remarks.

The clock–regulated queuing problems encountered in the information transmitting networks of state–of–the–art large scale digital computers are substantially more complex than analogous continuous time problems. Ultimately, one would want, at the very least, algebraic tools to deal with steady state which are as effective as those in the Poisson process context which suffices for the continuous case. In this paper, an extension of several in the literature, we have taken a first step, finding out how a single queue processes typical first stage inputs, and generalizing a portion of this to the case of queues of finite capacity. The corresponding treatment of later stages in such a network requires the use of more powerful mathematical tools, which are now the subject of investigation.

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References

[1] Feller, W. An Introduction to Probability Theory and Its Applications. Vol. I, Theorem 3, p. 244. John Wiley, New York, 1950.

[2] Gottlieb, A. An Overview of the N.Y.U. Ultracomputer Project, in: Experimental Parallel Computing Architectures, Dongarra, J. J. ed., pp. 25-95, 1987.

[3] Kruskal, C. P., Snir, M., and Weiss, A. The Distribution of Waiting Times in Clocked Multistage Interconnection Networks, Proceedings of the 1986 International Conference on Parallel Processing, pp. 12–19, 1986.

[4] Little, J. D. C. A Proof of the Queueing Formula $L = \lambda W$. Operations Research 9, pp. 383–387, 1961.

[5] Whittaker, E. T., Watson, G. N. A Course of Modern Analysis,Cambridge Univ. Press, Cambridge, 1962.

[6] Whittle, P., Systems in Stochastic Equilibrium, Part III,Network models, John Wiley, New York, 1986.

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